

# ON THE COMPLEXITY OF DIOPHANTINE GEOMETRY IN LOW DIMENSIONS (EXTENDED ABSTRACT)

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**ABSTRACT.** We consider the average-case complexity of some otherwise undecidable or open Diophantine problems. More precisely, we show that the following two problems can be solved in **PSPACE**:

- I. Given polynomials  $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$  defining a variety of dimension  $\leq 0$  in  $\mathbb{C}^n$ , find all solutions in  $\mathbb{Z}^n$  of  $f_1 = \dots = f_m = 0$ .
- II. For a given polynomial  $f \in \mathbb{Z}[v, x, y]$  defining an irreducible nonsingular non-ruled surface in  $\mathbb{C}^3$ , decide the sentence  $\exists v \forall x \exists y f(v, x, y) \stackrel{?}{=} 0$ , quantified over  $\mathbb{N}$ .

Better still, we show that the truth of the **Generalized Riemann Hypothesis (GRH)** implies that detecting roots in  $\mathbb{Q}^n$  for the polynomial systems in problem (I) can be done via a two-round Arthur-Merlin protocol, i.e., well within the second level of the polynomial hierarchy. (Problem (I) is, of course, undecidable without the dimension assumption.) The decidability of problem (II) was previously unknown. Along the way, we also prove new complexity and size bounds for solving polynomial systems over  $\mathbb{C}$  and  $\mathbb{Z}/p\mathbb{Z}$ . A practical point of interest is that the aforementioned Diophantine problems should perhaps be avoided in the construction of crypto-systems.

## 1. INTRODUCTION AND MAIN RESULTS

The negative solution of Hilbert's Tenth Problem [Mat70, Mat93] has all but dashed earlier hopes of solving large polynomial systems over the integers. However, an immediate positive consequence is the creation of a rich and diverse garden of hard problems with potential applications in complexity theory, cryptology, and logic. Even more compelling is the question of where the boundary to decidability lies.

From high school algebra we know that detecting roots in  $\mathbb{Q}$  (or  $\mathbb{Z}$  or  $\mathbb{N}$ ) for polynomials in  $\mathbb{Z}[x_1]$  is tractable. However, in [Jon82], Jones showed that detecting roots in  $\mathbb{N}^9$  for polynomials in  $\mathbb{Z}[x_1, \dots, x_9]$  is already undecidable. Put another way, this means that detecting a positive integral point on a general algebraic hypersurface of (complex) dimension 8 is undecidable.

It then comes as quite a shock that decades of number theory still haven't settled the complexity of the analogous question for algebraic varieties of dimension 1 through 7. In fact, even the case of plane curves remains a mystery:<sup>1</sup> As of late 1998, the decidability of detecting a root in  $\mathbb{N}^2$ ,  $\mathbb{Z}^2$ , or even  $\mathbb{Q}^2$ , for an **arbitrary** polynomial in  $\mathbb{Z}[x_1, x_2]$ , is still completely open.

**1.1. Dimension Zero.** We will thus go one dimension lower<sup>2</sup> in order to prove a useful result.

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<sup>1</sup>In particular, the major "solved" special cases so far have only extremely ineffective complexity and height bounds. (See, e.g., the introduction and references of [Roj99b].)

<sup>2</sup>We use the natural convention that  $\dim Z := -1$  when  $Z = \emptyset$ .

**Main Theorem 1.** *Suppose  $F := (f_1, \dots, f_m)$  is a system of polynomials in  $\mathbb{Z}[x_1, \dots, x_n]$  and let  $Z$  be the zero set of  $F$  in  $\mathbb{C}^n$ . Assume further that  $\dim Z \leq 0$ . Then*

1. *We can find all the roots of  $F$  in  $\mathbb{Z}^n$  within **PSPACE**.*
2. *The truth of the Generalized Riemann Hypothesis implies that deciding  $Z \cap \mathbb{Q}^n \stackrel{?}{=} \emptyset$  is in **AM**.*

**Remark 1.** *Recall that  $\mathbf{NP} \cup \mathbf{BPP} \subseteq \mathbf{AM} \subseteq \mathbf{RP}^{\mathbf{NP}} \cap \mathbf{coNP}^{\mathbf{NP}} \subseteq \Pi_2 \subseteq \dots \subseteq \mathbf{PH} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXPTIME}$  [BM88, Pap95]. Also, it will later follow easily (cf. remark 9) that the rational analogue of assertion (1) can be done within **EXPTIME** (or polynomial time for fixed  $n$ ).*

While one can derive assertion (1) more or less directly (from, say, [Ren87] or [BPR92]), the explicit sequential and parallel complexity bounds we give in section 3 (cf. remark 9) are the best to date and do **not** follow from earlier work. Also, assertion (2) presents a new arithmetic analogue of a recent result of Koiran [Koi96] stating that the truth of GRH implies that detecting a **complex** root can also be done within **AM**, with no restriction on  $\dim Z$ .

**Remark 2.** *Deciding  $\dim Z \stackrel{?}{\leq} 0$  can be done in **PSPACE** via another algorithm of Koiran [Koi97]. In fact, the truth of GRH implies that this decision problem can be done within **AM** as well [Koi97].*

**Remark 3.** *If one fixes the monomial term structure of  $F$  and picks the coefficients randomly, then it follows easily from the theory of resultants [GKZ94, Stu98] that  $m \geq n \implies F$  will have only finitely many roots in  $\mathbb{C}^n$  on average. This can be made completely explicit via, say, J. Schwartz' well-known result [Sch80] on randomized verification of polynomial identities.*

**Remark 4.** *We will use the classical Turing machine [Pap95] as our computational model, along with the bit-wise (resp. sparse) encoding for rational numbers (resp. polynomials in  $\mathbb{Z}[x_1, \dots, x_n]$ ) [BSS98], throughout. So, for example, the size of an integer  $c$  will be  $1 + \lceil \log_2(|c| + 1) \rceil$ . Also, all quantification symbols will be understood to range over the positive integers  $\mathbb{N}$ .*

An interesting corollary of Main Theorem 1 is the following:

**Corollary 1.** *Following the notation of Main Theorem 1, assume  $n$  is **fixed**. Then we can find all the roots of  $F$  in  $\mathbb{Z}^n$  within **NC<sub>3</sub>**. ■*

For  $n = 1$ , this complements two older results: (a) the famous result of Lenstra, Lenstra, and Lovasz [LLL82] that all the rational roots of  $f$  can be found in polynomial sequential time, and (b) Neff's result [Nef94] that the roots of  $f$  in  $\mathbb{C}$  can be approximated (to some input precision) in **NC<sub>3</sub>**. We have thus obtained higher-dimensional Diophantine analogues of these results.

The proof of Main Theorem 1 is based on the following number-theoretic result relating root-finding over the fields  $\mathbb{Q}$  and  $\mathbb{Z}/p\mathbb{Z}$ .

**Main Theorem 2.** *Following the notation of Main Theorem 1, let  $r$  be the number of roots of  $F$  in  $\mathbb{Q}^n$  counting multiplicities. Also define  $N_F(x)$  to be the **total** number of roots of  $F$  in  $\mathbb{Z}/p\mathbb{Z}$ , counting multiplicities, as  $p$  runs through all primes  $\leq x$ . Finally, let  $\pi(x)$  denote the number of primes  $\leq x$ . Then the truth of GRH implies that*

$$\left| \frac{N_F(x)}{\pi(x)} - r \right| \leq \frac{A}{\sqrt{x}} [C_F \log x + \mathcal{M}(E)(\log x)^2],$$

where  $A$  is an effective and absolute constant, and  $C_F$  is a quantity polynomial in  $S(\bar{E})$  and the size of  $F$ .

This result may be of independent interest to number theorists, as well as complexity theorists and numerical analysts. We also note that while Main Theorem 2 deals with using reduction mod  $p$  to count roots over  $\mathbb{Q}$ , other results, such as [Koi96, Thm. 8] and [Bür99, Thm. 4.1], use reduction mod  $p$  to determine the existence of roots in  $\mathbb{C}$ . Our techniques can be used to improve these latter results as well, and this will be pursued in a forthcoming paper.

**Remark 5.** *The main results we describe in the next two subsections are of a more technical nature, so the reader more interested in the conceptually simple **AM** algorithm above can skip directly to the beginning of section 3.*

**1.2. Dimensions One and Two.** To reconsider the complexity of detecting integral points on varieties of dimension  $\geq 1$ , one can consider more subtle combinations of quantifiers to facilitate finding decisive results. For example, Matiyasevich and Julia Robinson have shown [MR74, Jon81] that sentences of the form  $\exists u \exists v \forall x \exists y f(u, v, x, y) \stackrel{?}{=} 0$  are already undecidable.

However, the decidability of sentences of the form  $\exists v \forall x \exists y f(v, x, y) \stackrel{?}{=} 0$  was an open question until recently: in [Roj99b] it was shown that these sentences can be decided by a Turing machine, once the input  $f$  is suitably restricted. Roughly speaking, deciding the prefix  $\exists \forall \exists$  is equivalent to determining whether an algebraic surface has a slice (parallel to the  $(x, y)$ -plane) densely peppered with positive integral points. The “exceptional”  $f$  not covered by the algorithm of [Roj99b] form a very slim subset of  $\mathbb{Z}[v, x, y]$ .

We will further improve this result by showing that under similarly mild input restrictions,  $\exists \forall \exists$  can in fact be decided in singly exponential time and parallelized effectively. To make this more precise, let us write any  $f \in \mathbb{Z}[v, x, y]$  as  $f(v, x, y) = \sum c_a v^{a_1} x^{a_2} y^{a_3}$ , where the sum is over certain  $a := (a_1, a_2, a_3) \in \mathbb{Z}^3$ . We then define the **support** of  $f$  as  $\text{Supp}(f) := \{a \mid c_a \neq 0\}$ . The **Newton polytope** of  $f$ ,  $\text{Newt}(f)$ , is then just the convex hull<sup>3</sup>  $\text{Conv}(\text{Supp}(f))$ . When we say that a statement involving a set of parameters  $\{c_1, \dots, c_N\}$  is true **generically**, we will mean that the statement holds for all  $(c_1, \dots, c_N) \in \mathbb{C}^N$  outside of some **a priori fixed** algebraic hypersurface.

**Main Theorem 3.** *Fix the Newton polytope  $P$  of a polynomial  $f \in \mathbb{Z}[v, x, y]$  and let  $\rho$  denote the projection mapping  $\mathbb{R}^3$  onto the  $(\hat{e}_2, \hat{e}_3)$ -plane. Suppose further that  $\rho(P)$  has at least one lattice point in its interior. Then, for a generic choice of coefficients depending only on  $P$ , we can decide  $\exists v \forall x \exists y f(v, x, y) \stackrel{?}{=} 0$  within **PSPACE**.*

The generic choice above is clarified further in section 4. In particular, we will see that the “average” polynomial in  $\mathbb{Z}[v, x, y]$  with moderately large support defines an irreducible, nonsingular, non-ruled surface in  $\mathbb{C}^3$ .

It is interesting to note that the exceptional case to our above algorithm judiciously contains an extremely hard number-theoretic problem: determining the existence of a point in  $\mathbb{N}^2$  on an algebraic plane curve. (Indeed, we will see later that  $\mathbb{Z}[v, y]$  lies in our exceptional locus.) We also point out that the problem of computing the size of the **largest** positive integral point on an algebraic plane curve is closely related to determining whether an algebraic surface possesses **any** integral point: It was recently shown in [Roj99b] that under certain assumptions, the decidability of the latter problem implies the uncomputability of the former function.

Aside from a geometric trick, the proof of Main Theorem 2 relies on essentially the same tools as the proof of Main Theorem 1: Both proofs make use of the toric resultant [GKZ94, Stu98, Roj99c]

<sup>3</sup>That is, the smallest convex set in  $\mathbb{R}^3$  containing  $\text{Supp}(f)$ .

and a recent perturbation trick specifically tailored for degenerate sparse polynomial systems [Roj99c]. New complexity and size estimates on polynomial system solving over  $\mathbb{C}$  form a crucial key step. We now describe these new bounds.

**1.3. New Bounds for Solving Over  $\mathbb{C}$ .** In order to rigourously state our results for polynomial system solving over  $\mathbb{C}$ , we will introduce some necessary notation: First note that the notions of support and Newton polytope extends naturally to polynomials in  $\mathbb{Z}[x_1, \dots, x_n]$ . We then say that  $F$  is an  $\mathbf{m} \times \mathbf{n}$  **polynomial system with support contained in  $\mathbf{E}$** , whenever  $F$  is as stated in Main Theorem 1,  $E = (E_1, \dots, E_m)$ , and  $\text{Supp}(f_i) \subseteq E_i$  for all  $i$ .

An important geometric invariant for  $m \times m$  systems of equations is  $\mathcal{M}(E)$  — the **mixed volume** [BZ88, GK94, EC95, Ewa96, DGH98] of the convex hulls of  $E_1, \dots, E_m$ . A trick we will use to solve general  $m \times n$  systems (when  $m \geq n$ ) is to include additional points in the  $E_i$  so that  $\mathcal{M}(E) > 0$ .

For  $(m+1) \times m$  systems we will instead let  $E := (E_1, \dots, E_m)$  and  $\bar{E} := (E_1, \dots, E_{m+1})$ . We also point out the following two important complexity-theoretic parameters:

**Definition 1.** For any  $(m+1)$ -tuple  $\bar{E}$  of finite subsets of  $\mathbb{Z}^m$ , define  $\mathbf{R}(\bar{E}) := \sum_{i=1}^{m+1} \mathcal{M}(E_1, \dots, \widehat{E_i}, \dots, E_{m+1})$  and let  $\mathbf{S}(\bar{E}) = \mathcal{O}(\sqrt{m}e^m \mathcal{M}_{\bar{E}}^{\text{ave}})$ , where  $\mathcal{M}_{\bar{E}}^{\text{ave}}$  is the average value of  $\mathcal{M}(\mathcal{E})$  as  $\mathcal{E}$  ranges over all  $m$ -tuples  $(\mathcal{E}_1, \dots, \mathcal{E}_m)$  with  $\mathcal{E}_i \in \{E_1, \dots, E_{m+1}\}$  for all  $i$ .

While the precise definition of  $S(\bar{E})$  depends on the efficiency of a particular class of algorithms described in [Roj99c], the preceding asymptotic bound will suffice for our purposes.

**Remark 6.** An important extreme class of  $F$  is the **dense** case: This occurs when, for all  $i$ , all monomial terms up to some fixed degree occur in  $f_i$ . In this case, the best known complexity bounds for existential quantifier elimination over  $\mathbb{C}$  are polynomial in  $D_\Pi$  and  $\binom{D_\Sigma + 1}{n}$ , where  $D_\Pi$  and  $D_\Sigma$  are respectively the product and sum of the total degrees of the  $f_i$  [CKL89, Ier89, FGM90, Roj99c].

**Remark 7.** Our complexity and size bounds will instead be polynomial in the invariants  $\mathcal{M}(E)$ ,  $R(\bar{E})$ , and  $S(\bar{E})$ . In particular, we point out that  $\mathcal{M}(E) \leq R(\bar{E}) \leq S(\bar{E})$ . Furthermore,  $\mathcal{M}(E) \leq D_\pi$ ,  $R(\bar{E}) \leq (n+1)D_\Pi$ , and  $S(\bar{E}) \leq \binom{D_\Sigma + 1}{n} < (\frac{eD_\Sigma}{n})^n$ , with equality if the Newton polytopes are the same as those of the dense case. Even better, for sparse polynomial systems, our invariants are usually dramatically smaller than the last three upper limits (cf. example 2).

To bound the size of the roots of very general polynomial systems, we will need one final invariant.

**Definition 2.** Following the notation of definition 1, let  $c, k \in \mathbb{N}$  and define

$$H(c, k, \bar{E}) := 2^{\mathcal{M}(E)} (2m^2 \mathcal{M}(E)^2)^{m\mathcal{M}(E)} (c\sqrt{k})^{S(\bar{E}) - \mathcal{M}(E)}.$$

Let  $\Delta := \text{Conv}(\{\mathbf{O}, \hat{e}_1, \dots, \hat{e}_m\})$ , where  $\mathbf{O} \in \mathbb{R}^m$  denotes the origin and  $\hat{e}_i \in \mathbb{R}^m$  is the  $i^{\text{th}}$  standard basis vector. In what follows,  $\mathcal{O}^*(T)$  means  $\mathcal{O}(T \log^r T)$  for some constant  $r > 0$ .

**Main Theorem 4.** Following the notation and hypotheses of Main Theorem 1, suppose further that  $m \geq n$ , every  $f_i$  has at most  $k$  monomial terms, and that all the coefficients of  $F$  have absolute value  $\leq c$ . Also, for all  $i \in \{1, \dots, m\}$  define  $E_i$  to be the union of  $\{\mathbf{O}, \hat{e}_i\}$  and the support of  $f_i$ . Finally, let  $E_{m+1} = \Delta \cap \mathbb{Z}^m$ . Then we can find univariate polynomials  $h, h_1, \dots, h_n \in \mathbb{Z}[t]$  with the following properties:

0. The degrees of  $h, h_1, \dots, h_n$  are bounded above by  $\mathcal{M}(E)$ .

1. Given any root  $\theta \in \mathbb{C}$  of  $h$ , define  $\gamma(\theta) := (h_1(\theta), \dots, h_n(\theta))$ . Then the set of points  $\{\gamma(\theta)\}_{h(\theta)=0}$  contains all the roots of  $F$  in  $\mathbb{C}^n$ .
2. The coefficients of  $h$  have absolute value bounded above by  $H(c, k, \bar{E})$

In particular, the size of the coefficients of  $h$  is polynomial in  $S(\bar{E})$  and  $L_F$ , where  $L_F$  denotes the size of  $F$ . Furthermore,  $h, h_1, \dots, h_n$  can be found deterministically within **sequential** time  $\mathcal{O}^*(nm^3\mathcal{M}(E)^3S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\}L_F)$ , or **parallel** time  $\mathcal{O}(\log\{S(\bar{E})L_F\})$  using  $\mathcal{O}^*(nm^3\mathcal{M}(E)^3S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\})$  processors. In particular,  $h, h_1, \dots, h_n$  can be found in **NC** when  $n$  is fixed.

**Remark 8.** Choosing extra points to add to  $\text{Supp}(F)$  (and the choice of Newton polytope for  $f_{m+1}$ ) can be optimized for a given monomial term structure without too much difficulty (see [Roj99c, remark 2]).

The complexity bounds above are the best deterministic bounds to date for **general**<sup>4</sup> univariate reduction over  $\mathbb{C}$ . In particular, our bounds work for over-determined systems of equations and are much more sensitive to the sparse encoding of the input than earlier bounds (cf. example 2). For instance, our bounds above considerably improve those of [Ier89, FGM90], which were stated solely in terms of the degrees of the input polynomials.<sup>5</sup> Furthermore, the exponents above can be significantly lowered if randomization is allowed [Roj99c].

As an almost immediate corollary, we obtain new and extremely general bounds on the size of the roots of  $F$ :

**Main Theorem 5.** Following the notation and hypotheses of Main Theorems 1 and 4, let  $(x_1, \dots, x_n) \in \mathbb{C}$  be any root of  $F$ , fix any  $j \in \{1, \dots, n\}$ , and now let  $E_{m+1}$  be the line segment from the origin  $\mathbf{O}$  to  $\hat{e}_j$ . Then either  $x_j = 0$  or  $H(c, k, \bar{E})^{-1} \leq |x_j| \leq H(c, k, \bar{E})$ . In particular, the size of the roots of  $F$  is polynomial in  $S(\bar{E})$  and the size of  $F$ .

The above gap theorem generalizes earlier bounds due to Malajovich-Muñoz [Mal94] and Canny [Can87]. In particular, while our bound is a bit coarser in the dense case, it is significantly better for certain sparse systems (cf. example 2). Furthermore, earlier bounds assumed  $m = n$  and made various other nondegeneracy assumptions — such as no multiple roots — even in the case  $m = n = 1$ .

Main Theorems 1, 3, 4, and 5 are respectively proved in sections 3, 4, 5, and 6. Section 6 also contains the proof of Main Theorem 2. However, we will first give some examples before starting our main proofs.

## 2. ALGORITHMIC EXAMPLES

We begin with a brief illustration and synopsis of the algorithm from Main Theorem 4. We then conclude with an example of the benefits of the sensitivity to sparsity (or “monomial sensitivity”) in our complexity and size bounds.

Further details on the univariate reduction algorithm below appear in section 5, and the case  $m = n$  is described at length in [Roj99c].

### Example 1. ( $m = n = 2$ )

Consider the bivariate polynomial system  $F = (1 + 2x - 2x^2y - 5xy + x^2 + 3x^3y,$

<sup>4</sup>The algorithm above can easily be generalized to the case  $\dim Z \geq 1$  without changing the complexity bounds, via the techniques of [Roj99c].

<sup>5</sup>We point out, however, that these papers deal with a more general problem than Main Theorem 4.

$2 + 6x - 6x^2y - 11xy + 4x^2 + 5x^3y$ ). Letting  $E$  be the support of  $F$ , the reader can easily verify<sup>6</sup> that  $\mathbf{O} \in E_1 \cap E_2$ ,  $\mathcal{M}(E) = 4$ , and that the only roots of  $F$  are the points  $\{(1, 1), (\frac{1}{7}, \frac{7}{4})\}$  and the line  $\{-1\} \times \mathbb{C}$ . So this example illustrates a slightly more general situation than our  $\dim Z = 0$  restriction.

We now highlight the main points of our algorithm: First, via combinatorial means [EC95], we construct a **toric resultant matrix**,  $M_{\bar{E}}$ . The entries of this sparse and highly-structured matrix (of size  $S(\bar{E}) \times S(\bar{E})$ ) will be the coefficients of  $F$ , as well as a few extra parameters.

The coefficients of the polynomials  $h, h_1, h_2$  are found by solving a collection of linear systems of equations. The coefficients for these systems are in turn derived from the computation of determinants of various specializations of  $M_{\bar{E}}$ . The number of linear systems and determinants, as well as their sizes, accounts for the complexity bound of Main Theorem 4. Note in particular that  $R(\bar{E}) = 4 + 4 + 4 = 12$  and that  $S(\bar{E})$  can be taken to be 17 in the case at hand. (The corresponding matrix  $M_{\bar{E}}$  appears explicitly in [Roj99c].)

After an application of **Maple**, one at last obtains that

$$h(t) = -153 + 120t + 1540t^2 + 1600t^3 + 448t^4,$$

$$h_1(t) = \frac{-11762}{7511} + \frac{19150}{22533}t + \frac{114736}{22533}t^2 + \frac{7264}{3219}t^3, \text{ and } h_2(t) = \frac{-5881}{7511} + \frac{32108}{22533}t + \frac{57368}{22533}t^2 + \frac{3632}{3219}t^3.$$

Since  $h(t)$  factors as

$$(2t + 3)(28t + 51)(2t + 1)(4t - 1),$$

we immediately obtain a set of points lying in  $Z$  (including all the isolated roots) by substituting  $\{-\frac{3}{2}, -\frac{51}{28}, -\frac{1}{2}, \frac{1}{4}\}$  into the pair  $(h_1(t), h_2(t))$ . Note also that the roots of  $h$  are exactly  $-\frac{1}{2}\zeta_1 - \zeta_2$ , as  $(\zeta_1, \zeta_2)$  ranges over a finite set of roots of  $F$ , at least one in each irreducible component of  $Z$ .

Our next example shows how older complexity and size bounds (which were stated solely in terms of degrees of polynomials) may be too pessimistic for certain sparse systems.

**Example 2. (Well Directed Spikes)** Consider the system of equations  $F$  defined by

$$\begin{aligned} & a_{1,1} + a_{1,2}x_1 + \cdots + a_{1,n}x_{n-1} \\ & + c_{1,1}(x_1 \cdots x_n) + \cdots + c_{1,d}(x_1 \cdots x_n)^d = 0 \\ & \vdots \\ & a_{n,1} + a_{n,2}x_1 + \cdots + a_{n,n}x_{n-1} \\ & + c_{n,1}(x_1 \cdots x_n) + \cdots + c_{n,d}(x_1 \cdots x_n)^d = 0. \end{aligned}$$

In this case, the Newton polytopes are all equal to a single “spike” and we can actually pick the Newton polytope of  $f_{n+1}$  to be  $\text{Conv}(\{\mathbf{O}, \hat{e}_1, \dots, \hat{e}_{n-1}, \sum \hat{e}_j\})$ . So via the basic properties of the mixed volume, it is a routine exercise to check that  $\mathcal{M}(E) = d$  and  $R(\bar{E}) = (n+1)d$ . Furthermore, note that our example is equivalent (via the change of coordinates  $y = (x_1, \dots, x_{n-1}, x_1 \cdots x_n)$ ) to a system of total degree 1 in  $y_1, \dots, y_{n-1}$  and of degree  $d$  in  $y_n$ . It then follows from [GKZ94, Roj99c] that we can build a sufficiently compact resultant matrix so that  $S(\bar{E}) = R(\bar{E})$ . Main Theorem 4

<sup>6</sup>For  $n = 2$ , there is the simple formula  $\mathcal{M}(E) = \text{Area}(\text{Conv}(E_1 + E_2)) - \text{Area}(\text{Conv}(E_1)) - \text{Area}(\text{Conv}(E_2))$ . Also, both polynomials are divisible by  $x + 1$ .

thus tells us that we can reduce solving  $F$  to a univariate problem within deterministic sequential time  $\mathcal{O}^*(n^{7.376}d^{6.376})$ .

Let us now see how previous methods fare on our sparse example. Remark 6 tells us that older algorithms took sequential time polynomial in  $D_\Pi$  and a rather large binomial coefficient. These two parameters respectively reduce to  $n^n d^n$  and  $\Omega(n^n (\frac{d}{e})^n)$  (by Stirling's estimate). So our methods clearly exploit sparsity to much greater advantage.

As for root size bounds, the best previous bound for dense systems of **identical** degrees [Can87] specializes to  $(3cdn)^{d^n n^{n+1}}$ . Roughly speaking, this means that this older bound needs  $\mathcal{O}(bd^n n^{n+1})$  bits to specify the size of the roots of  $F$ , assuming the coefficients of  $F$  used  $b$  bits to start with. On the other hand, our new bound from Main Theorem 5 tells us that we need only  $\mathcal{O}(nd(b + \log(nd)))$  bits. So it appears that sparsity is also quite helpful for root size bounds.

Generating infinite families of such examples is easy, simply by picking Newton polytopes which are  $n$ -dimensional, but “long” in a suitable fixed direction.

### 3. THE PROOF OF MAIN THEOREM 1

We will first present a proof of assertion (2), since the corresponding algorithm is much simpler than our more intricate **PSPACE** algorithm.

**Proof of Assertion (2):** Consider the following algorithm:

**Step 1** Pick positive integers  $s$  and  $t$  as follows: Let  $t$  be just large enough so that

$$\frac{A}{\sqrt{t}}[C_F \log t + \mathcal{M}(E)(\log t)^2] < \frac{1}{4}$$

and  $t > 4n\mathcal{M}(E)$ . Then define  $s$  to be just small enough so that  $\frac{t}{s} > \frac{\pi(t)}{4}$ .

**Step 2** Pick a (uniformly) random integer in  $j \in \{1, \dots, \lceil \frac{t}{s} \rceil - 1\}$  and define  $x_j$  and  $x_{j+1}$  to be  $js$  and  $(j+1)s$  respectively. (If  $j+1 = \lceil \frac{t}{s} \rceil$  then define  $x_{j+1} := t+1$ .)

**Step 3** Via an **NP** oracle, find if there is a prime  $p \in [x_j, x_{j+1})$  such that  $F$  has a solution in  $\mathbb{Z}/p\mathbb{Z}$ .

**Step 4** If such a prime exists, conclude that  $F$  has a root in  $\mathbb{Q}^n$ . Otherwise, declare that there is no root in  $\mathbb{Q}^n$ .

This algorithm clearly at least resembles a two-round Arthur-Merlin protocol, so let us confirm its correctness.

First note if  $F$  has a root in  $\mathbb{Q}^n$ , then  $F$  has a root in  $\mathbb{Z}/p\mathbb{Z}$  for all but finitely many primes  $p$ . (The exceptional primes must divide the denominator of some coordinate of a rational root of  $F$ , so there can be at most  $n\mathcal{M}(E)$  such primes.) So in this case, by our choice of  $t$ ,  $F$  will have a root in  $\mathbb{Z}/p\mathbb{Z}$  for more than  $\frac{3}{4}$  of the primes  $p \in [1, t+1)$ . So, by our choice of  $s$ , our algorithm has a probability greater than  $\frac{3}{4}$  of succeeding in this case. The role of the size of  $s$  is detailed further in the next case:

Suppose now that  $F$  has no roots in  $\mathbb{Q}^n$ . Call a prime for which  $F$  has a root in  $\mathbb{Z}/p\mathbb{Z}$  a **bad** prime. Then by Main Theorem 2 and our choice of  $t$ , strictly less than  $\frac{1}{4}$  of the primes  $p \in [1, t+1)$  are bad. Also, by the box-principle, the probability that the interval  $[x_j, x_{j+1})$  contains a bad prime is less than  $\frac{1}{4}$  (thanks to our choice of  $s$ ). So the probability of failure in this case is strictly less than  $\frac{1}{4}$ .

To conclude, recall that  $\log S(\bar{E})$  is polynomial in the size of  $F$ . So the number of bits necessary to express any prime above is polynomial in the size of  $F$ . Also note that assuming GRH,  $\pi(x)$  (which is asymptotic to  $\frac{x}{\log x}$  [Wei84]) can be approximated within a factor of  $1 + \varepsilon$  in time polynomial in

$\frac{1}{\varepsilon}$  and the size of  $x$  [LO77]. (Furthermore, we can certainly compute an integer at least as large as  $S(\bar{E})$  in polynomial time via remark 7.) Thus, the time needed to compute  $\{x_j, x_{j+1}\}$ , the number of necessary random bits, and the number of bits of any integer in  $[x_j, x_{j+1})$ , are all polynomial in the size of  $F$ .

Finally, via [Mil75], the truth of GRH implies that an integer  $p \in [x_j, x_{j+1})$  can be verified to be a prime in polynomial time. So via [Coh93], a putative root of  $F \bmod p$  can indeed be verified in polynomial time. So we indeed need only one call to an **NP** oracle. Our algorithm is thus indeed an **AM** algorithm. ■

**Proof of Assertion (1):** First note that if  $m < n$  then  $Z \neq \emptyset \implies \dim Z \geq 1$ , by the well-known facts on intersections of complex hypersurfaces [Mum95]. So we can safely assume that  $m \geq n$ .

Following the notation of the proof of Main Theorem 4, now pick  $f_{m+1} = u_0 + u_i x_i$  and run the algorithm from Main Theorem 4 (cf. remark 15) to solve for the  $i^{\text{th}}$  coordinates of all the roots of  $F$  within accuracy  $\frac{1}{3}$ . By Main Theorem 4, and combining with Neff's algorithm for **NC** univariate solving over  $\mathbb{C}$  [Nef94], this computation can be done for all  $i \in \{1, \dots, n\}$  within **PSPACE**. (Note that we are implicitly using the fact that polynomial parallel time with singly exponentially work is in **PSPACE** [Pap95, pg. 398].)

By taking combinations of the coordinates thus found, we obtain a set  $R \subset \mathbb{C}^n$  of at most  $\mathcal{M}(E)^n$  putative approximate solutions of  $F$ . In particular, by the choice of accuracy we've made, the integral roots of  $F$  are contained in the set  $R' := \{([x_1], \dots, [x_n]) \mid (x_1, \dots, x_n) \in T\}$ , where  $[x]$  denotes the nearest integer to  $x$ . The integral roots of  $F$  then certainly lie in  $R'$ . They can then be identified within **PSPACE**, using the most naive parallel algorithm for polynomial evaluation, devoting at worst polynomially many processors to each element of  $R'$ . The quantity  $\mathcal{M}(E)$  is at worst singly exponential in the coordinates of the  $E_i$  (cf. remark 7), so we are done. ■

**Remark 9.** A more precise complexity bound for algorithm above is the following:

$$\mathcal{O}^*(n^2 m^3 \mathcal{M}(E)^3 S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\} L_F)$$

sequential time (via fast approximate univariate factorization over  $\mathbb{C}$  [BP94]) or  $\mathcal{O}^*(\log^3\{S(\bar{E})L_F\})$  parallel time using  $\mathcal{O}^*(nm^3 \mathcal{M}(E)^3 S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\})$  processors. (So we obtain **NC**<sub>3</sub> for fixed  $n$ .) Furthermore, it easy to see that we can substitute the factoring algorithm of [LLL82], in place of Neff's algorithm, to find all the roots of  $F$  in  $\mathbb{Q}^n$  with **EXPTIME**. For fixed  $n$  this clearly reduces to polynomial time.

**Remark 10.** Presumably, Main Theorem 1 continues to hold under the weaker condition that the **real** dimension of  $Z$  is at most zero. One route of proof is an extension of Main Theorem 5 to the **real** isolated roots of  $F$ , and this will be pursued in later work.

#### 4. GENUS ZERO VARIETIES AND THE PROOF OF MAIN THEOREM 3

In what follows, we will make use of some basic algebraic geometry. A more precise description of the tools we use can be found in [Roj99b]. Also, we will always use **geometric** (as opposed to arithmetic) genus for algebraic varieties [Har77].

Let us begin by clarifying the genericity condition of Main Theorem 3. Let  $Z$  be the zero set of  $f$ . What we will actually require of  $f$  (in addition to the assumptions on its Newton polytope) is that  $Z$  be an irreducible nonsingular surface of positive genus. That  $Z$  is irreducible and nonsingular for a generic choice of coefficients follows from Bertini's theorem [Mum95]. That  $Z$  also has positive genus generically follows from a result of Khovanskii [Kho78]. (His result actually implies that



for generic coefficients and generic  $v_0$ ,  $f(v_0, x, y) = 0$  defines a curve of positive genus. It is then impossible for  $Z$  to generically have genus zero.) Since the intersection of any two open Zariski-dense sets is open and dense, we indeed have that our hypothesis occurs generically.

Now note that from the classification of algebraic surfaces [Bea96],  $Z$  has positive genus  $\implies Z$  is non-ruled. In particular, this means that there can only be finitely many  $v_0$  such that the “slice”  $Z \cap \{v = v_0\}$  contains a curve of genus zero. (By the nonsingularity of  $Z$  and Bertini’s theorem again.) Note that by Siegel’s Theorem [Sil99],  $\forall x \exists y f(v_0, x, y) = 0 \implies Z \cap \{v = v_0\}$  contains a curve of genus zero. So assuming one can decide the prefix  $\forall\exists$ , finding genus zero slices for the large family of  $f$  above gives us a way to decide the prefix  $\exists\forall\exists$ .

The preceding assumption is true, in spades, thanks to the following result:

**The JST Theorem.** [Jon81, Sch82, Tun87] *The quantifier prefix  $\forall\exists$  is decidable in sequential time polynomial in  $\deg f$  and singly exponential in the size of  $f$ . More explicitly, given  $f \in \mathbb{Z}[x, y]$ , we have that  $\forall x \exists y f(x, y) = 0$  iff all of the following conditions hold:*

1. *The polynomial  $f$  factors into the form  $f_0(x, y) \prod_{i=1}^k (y - f_i(x))$  where  $f_0(x, y) \in \mathbb{Q}[x, y]$  has **no** zeroes in the ring  $\mathbb{Q}[x]$ , and for all  $i$ ,  $f_i \in \mathbb{Q}[x]$  and the leading coefficient of  $f_i$  is positive.*
2.  *$\forall x \in \{1, \dots, x_0\} \exists y \in \mathbb{N}$  such that  $f(x, y) = 0$ , where  $x_0 = \max\{s_1, \dots, s_k\}$ , and for all  $i$ ,  $s_i$  is the sum of the squares of the coefficients of  $f_i$ .*
3. *Let  $\alpha$  be the least positive integer such that  $\alpha f_1, \dots, \alpha f_k \in \mathbb{Z}[x]$  and set  $g_i := \alpha f_i$  for all  $i$ . Then the **union** of the solutions of the following  $k$  congruences*

$$\begin{aligned} g_1(x) &\equiv 0 \pmod{\alpha} \\ &\vdots \\ g_k(x) &\equiv 0 \pmod{\alpha} \end{aligned}$$

*is all of  $\mathbb{Z}/\alpha\mathbb{Z}$ . ■*

**Remark 11.** *The JST Theorem can be strengthened slightly in the following way: one can replace  $\alpha$  in condition (3) with **any** positive integer  $\alpha'$  such that  $\alpha' f_1, \dots, \alpha' f_k \in \mathbb{Z}[x]$ . Also, the techniques of [Coh93] can easily support the stated complexity bound.*

**Proof of Main Theorem 3:** It follows easily from the Hurwitz genus formula for curves [Sil95] that  $\forall x \exists y f(v_0, x, y) = 0 \implies Z \cap \{v = v_0\}$  defines a curve with a singular component. The set of such  $v_0 \in \mathbb{C}$  is of course finite (by Bertini’s theorem again), since  $Z$  was assumed to be nonsingular.

So our algorithm is the following: Find those  $v_0 \in \mathbb{N}$  for which  $Z \cap \{v = v_0\}$  is singular, and then solve the corresponding instances of  $\forall\exists$ . If any instance is true, then our original sentence was true. Otherwise, our original sentence was false.

Finding this set of  $v_0$  is easily done within polynomial sequential time using the Jacobian criterion for singularity [Mum95]: simply find those positive integers  $v_0$  for which the system of equations  $(f(v_0, x, y), \frac{\partial f(v_0, x, y)}{\partial x}, \frac{\partial f(v_0, x, y)}{\partial y})$  has a solution  $(x, y) \in \mathbb{C}^2$ . This can be done easily via Main Theorem 4. In particular, the number of eligible  $v_0$  is polynomial in the degree of  $f$ . (Polynomial in  $\text{Vol}(P)$  in fact). Furthermore, we can simply solve to within accuracy  $\frac{1}{3}$  to isolate the  $v_0 \in \mathbb{N}$  (if any).

To conclude, we simply note that the only part of the JST theorem which can not be implemented in polynomial sequential time, via the results we’ve introduced and quoted so far, is part (2). For this part we then simply use (singly) exponentially many processors (one for each  $x \in \{1, \dots, x_0\}$ )

to finish in constant additional time. Since  $\mathbf{P}$  (and constant parallel time with singly exponential work) is contained in  $\mathbf{PSPACE}$  [Pap95, pg. 398], we are done. ■

**Remark 12.** *Although a result weaker than Main Theorem 4 would have sufficed, an immediate corollary of our proof is that the parallel time sufficient to decide  $\exists\forall\exists$  is near-heptic in the volume of the Newton polytope  $P$ .*

**Remark 13.** *Note that if  $f \in \mathbb{Z}[v, y]$  then  $Z$  is a ruled surface in  $\mathbb{C}^3$ . From another point of view, the hypothesis of Main Theorem 3 is violated since  $\rho(P)$  is contained in a line segment. Deciding  $\exists\forall\exists$  for this case then reduces to deciding  $\exists\exists$ , which we've already observed is very hard. Nevertheless, Alan Baker has conjectured that this problem is decidable [Jon81, section 5].*

**Remark 14.** *The complexity of deciding whether a given surface is ruled is an open problem. (Although one can check certain instances in  $\mathbf{PSPACE}$ , as described above.) It is also interesting to note that finding explicit parametrizations of **rational** surfaces (a special class of ruled surfaces) appears to be decidable. Evidence is provided by an algorithm of Josef Schicho which, while still lacking a termination proof, seems to work well in practice [Sch98].*

## 5. DETERMINANTS GALORE AND THE PROOF OF MAIN THEOREM 4

Let us first concentrate on an important special case.

**The Case  $m=n$ :** This special case of our result, save for the parallel time and coefficient bounds, reduces to Main Theorem 1 of [Roj99c]. Note in particular that the sequential time bound for this case simplifies to  $\mathcal{O}^*(n^4 \mathcal{M}(E)^3 S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\} L_F)$ . There is also the issue of converting arithmetic cost to bit cost, so this is how the factor of  $L_F$  (not present in [Roj99c]) appears here. We will soon see that the structure of our algorithm permits a relatively simple conversion between these two different costs via, say, the techniques of [BP94].

We will now prove the parallel complexity bound, illustrating our algorithm along the way. What we will describe is essentially an outline of an algorithm detailed further in [Roj99c]:

- Step 0** Compute the toric resultant matrix  $M_{\bar{E}}$  corresponding to the support  $\bar{E}$ .
- Step 1** Upon suitably specializing  $u_1, \dots, u_m$  to generic integers, and picking a generic polynomial system  $F^*$  with support contained in  $E$ , define<sup>7</sup>  $\mathcal{H}(s, u_0)$  to be  $\det M_{\bar{E}}$ .
- Step 2** Let  $h(t)$  be the coefficient of the term of  $\mathcal{H}(s, t)$  of lowest degree in  $s$ .
- Step 3** Compute  $h_1, \dots, h_n$  by a slightly more complicated combination of interpolation and specialized determinants.

(The toric resultant  $\text{Res}_\star(\cdot)$  appears as follows:  $\det M_{\bar{E}}$  is actually a multiple of  $\text{Res}_{\bar{E}}(\cdot)$  evaluated at the polynomial system  $(F, f_{m+1})$ , where  $f_{m+1} = u_0 + u_1 x_1 + \dots + u_m x_m$ .)

Step (0) is a preprocessing step with complexity dominated by the remainder of our algorithm. The proof follows easily from the theory of [GKZ94, EC95] (via the Cayley trick, the well-known algorithms for simplicial subdivisions [PS85], and the mixed-subdivision algorithm for computing  $M_{\bar{E}}$ ), but would be too great a digression to include in this abstract. So we consider our remaining steps.

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<sup>7</sup>In reality,  $\mathcal{H}$  will be a divisor of a variant of this determinant. This accounts for the copious amount of interpolation below. See [Roj99c] for further details.

The determination of a generic  $F^*$  in Step (1) can be done in work well-dominated by the remainder of our algorithm, by Main Theorem 3 of [Roj99c]. (The parallelization of finding  $F^*$  is quite straightforward.)

As for the rest of Steps (1) and (2), a lucky choice of  $u_1, \dots, u_n$ , would reduce our work to evaluating  $\mathcal{O}(n\mathcal{M}(E) \min\{R(\bar{E})^2, S(\bar{E})\})$  determinants of size  $\mathcal{O}^*(S(\bar{E})) \times \mathcal{O}^*(S(\bar{E}))$  and one evaluation/interpolation problem of size  $\mathcal{O}^*(S(\bar{E}))$ . In the absence of such luck, one can instead be deterministic and evaluate  $\mathcal{O}^*(n^3\mathcal{M}(E)^3 \min\{R(\bar{E})^2, S(\bar{E})\})$  such determinants and solve  $\mathcal{O}^*(\mathcal{M}(E)^2)$  such interpolation problems. (Via a technique of [Roj99c, section 5.2], this results in a choice of  $u_i$  with absolute values at most  $\mathcal{O}(2^n n^{2n} \mathcal{M}(E)^{2n})$ .) In particular, it easily follows from recent parallel matrix algorithms, e.g., [BP94], that Steps (1) and (2) take overall (arithmetic) parallel time  $\mathcal{O}(\log S(\bar{E}))$  using  $\mathcal{O}^*(n^3\mathcal{M}(E)^3 S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\})$  processors. In terms of the Turing (bit) model, [BP94] (and the structure of our algorithm) tells us that we need only include an additional factor of  $L_F$ . So the overall parallel time is  $\mathcal{O}(\log\{S(\bar{E})L_F\})$  with the same number of processors.

By [Roj99c] it then follows that Step (3) amounts to  $n$  repetitions of what we just did for Step (2), and this work can be done in parallel. We thus at last arrive at a parallel complexity bound of  $\mathcal{O}(\log\{S(\bar{E})L_F\})$  time using  $\mathcal{O}^*(n^4\mathcal{M}(E)^3 S(\bar{E})^{2.376} \min\{R(\bar{E})^2, S(\bar{E})\})$  processors.

We now analyze the size of the coefficients of  $h$ . Here, we will consider a fundamental special case. The proof of the general case follows easily from the special case via known bounds on the coefficients of factors of multivariate polynomials, e.g., [Mig92].

So let us make the following assumption: the determinant of  $M_{\bar{E}}$  does not vanish identically.<sup>8</sup> In which case, it indeed suffices to define  $\mathcal{H}(s, u_0)$  as  $\det M_{\bar{E}}$  itself.

By construction, there will be exactly  $\mathcal{M}(E)$  rows of  $M_{\bar{E}}$  involving coefficients of  $f_{m+1}$  and exactly  $S(\bar{E}) - \mathcal{M}(E)$  rows involving the parameter  $s$ . Furthermore, the coefficients of  $F^*$  can all be assumed to be 1, by Main Theorem 3 of [Roj99c]. So by expanding  $\mathcal{H}(s, u_0)$  in terms of minors, and using Hadamard's inequality [Mig92], it easily follows that the coefficient of  $t^i$  in  $h(t)$  has absolute value at most  $\binom{\mathcal{M}(E)}{i} (2n^2\mathcal{M}(E)^2)^{n\mathcal{M}(E)} (c\sqrt{k})^{S(\bar{E})-\mathcal{M}(E)}$ . So by Stirling's formula we can conclude the case  $m=n$ . ■

**Remark 15.** *The main trick in the above algorithm is to use a generic linear form to project the roots of  $F$  onto  $\mathbb{C}$ . This is done so as to leave the underlying coordinate rings isomorphic over  $\mathbb{Q}$ . This motivated our choice of  $f_{m+1}$  above. In particular, the roots of  $h$  are exactly the image of the roots of  $F$  under the chosen linear form. However, this technique is quite general and other choices of  $f_{m+1}$  are possible and quite useful. For instance, picking  $f_{m+1} = u_0 + u_i x_i$  amounts to projecting the roots onto the  $x_i$ -axis.*

**The case  $m > n$ :** We first note that the zero set of  $F$ , now considered as an  $m \times m$  polynomial system, consists of a set of disjoint linear subvarieties. Under the natural embedding of  $\mathbb{C}^n \hookrightarrow \mathbb{C}^m$  (into the first  $n$  coordinates), these linear subspaces intersect  $\mathbb{C}^n$  precisely in the roots of  $F$ . Furthermore, these linear subspaces never intersect within the toric compactification  $\mathcal{T}$  corresponding to the polytope  $\sum_{i=1}^{m+1} \text{Conv}(E_i)$ . In particular, there is a one to one correspondence between these linear subspaces of  $\mathcal{T}$  and the points of  $Z$ .

<sup>8</sup>This assumption can be removed at the price of an additional evaluation/interpolation step. The details will be covered in the full version of this paper.

To solve  $F$ , it thus suffices to find a point in every irreducible component of the zero set of  $F$  in  $\mathcal{T}$ , and then project these points onto  $\mathbb{C}^n$ . Main Theorem 1 of [Roj99c] can do exactly this for us. As for the complexity and coefficient analysis, this proceeds almost the same as the case  $m=n$  solved above. The only exception is that we will need only  $n$  of the polynomials  $h_1, \dots, h_m$ . Our complexity and size bounds then follow immediately from the  $m=n$  case. ■

## 6. COEFFICIENT BOUNDS AND PRIME DISTRIBUTION: PROVING MAIN THEOREM 5 AND MAIN THEOREM 2

**Proof of Main Theorem 5:** The bounds on the coefficients of  $h$  derived in the proof of Main Theorem 4 (which are more precise than those stated in the theorem itself) are general enough that we can vary  $u_1, \dots, u_n$  to our advantage. In particular, setting all  $u_j$  to 0 except for  $u_i$ , we obtain that any  $i^{\text{th}}$  coordinate of a root of  $F$  must be a root of  $h$ . By [Mig92, theorem 4.2, (viii)], our root size bound then follows immediately. ■

**Proof of Main Theorem 2:** By Main Theorem 4 (see also remark 15), we obtain a polynomial  $h$  with the following properties: (a)  $\deg h \leq \mathcal{M}(E)$ , (b) the coefficients of  $h$  are integers of size polynomial in  $S(\bar{E})$  and the size of  $F$ , and (c)  $F$  has a rational root iff  $h$  has a rational root.

Main Theorem 2 then follows immediately from the main theorem of [Wei84], which is essentially just a univariate version of Main Theorem 2. In particular, our  $A$  is the same  $A$  as that of Weinberger, and the quantity  $C_F$  is just the logarithm of the discriminant of  $h$ . By Main Theorem 5 (and an application of the Jacobian),  $C_F$  has size polynomial in  $S(E)$  and the size of  $F$ . ■

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